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THE STABILITY OF THE PERIODIC STATIONARY STOKES EQUATIONS ON R^n

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Abstract. In this paper, we will consider the periodic stationary Stokes equations on R^n. For the cube of the period, we set $\Omega = \prod_{i=1}^{n}(0, L_i)$. And we will study the stability of the solutions on various functional spaces, for the Stokes equations on R^n.

1. Introduction

Obłoza [12, 13] studied the Hyers-Ulam stability of the linear differential equations, $y'(x) + g(x)y(x) = r(x)$. And it was based on Ulam's idea for the stability of homomorphism. Thereafter, this subject for different types of differential equations was discussed by many mathematicians. Consider an open interval $I = (a, b)$ of $\mathbb{R}$ with $-\infty \le a < b \le +\infty$. And, for an n times continuously differentiable function, $y : I \to \mathbb{C}$, let the linear differential equation,

$$\mathfrak{F}\big(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\big) = 0 \tag{1.1}$$

satisfies on I. For any $\varepsilon > 0$, we assume that there exists a solution $y_0 : I \to \mathbb{C}$ to (1.1) such that

$$|y(x) - y_0(x)| \le K(\varepsilon), \qquad \text{for all } x \in I, \tag{1.2}$$

where $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$, if $y : I \to \mathbb{C}$ satisfies the differential inequality

$$\big|\mathfrak{F}\big(y^{(n)}, y^{(n-1)}, \ldots, y', y, x\big)\big| \le \varepsilon. \tag{1.3}$$

Then we say that the differential equation (1.1) satisfies the Hyers-Ulam stability. One can find many interesting results for the linear differential equations from the following references, [3, 4, 5, 8, 9, 11, 14, 17]. Also, for the partial differential equations, one can refer to [1, 2, 6, 7, 10, 15].

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In this paper, we will study the stability of the solutions for the stationary Stokes equations on R^n, with the periodic boundary condition. For the stability of the periodic stationary Stokes equations, Roh-Jung[10] investigated for R^2. They used $\Omega = (0, L) \times (0, L)$ for the cube of the period. In this paper, we will use more general cube $\Omega = \prod_{i=1}^{n}(0, L_i)$ of the period.

We will consider the stationary Stokes problem with the following form: *For a given* $\mathbf{f}$*, we want to find* $\mathbf{u}$ *and* p *such that*

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \textit{in} \quad R^n \tag{1.4}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \textit{in} \quad R^n$$

$$\mathbf{u}(x + L_i e_i) = \mathbf{u}(x) \quad \textit{for all} \quad x \in R^n, \tag{1.5}$$

where $\{e_1, \dots, e_n\}$ *is the standard basis of* R^n*,* L_i *is the period in the* i*-th direction and* $\Omega = \prod_{i=1}^{n}(0, L_i)$ *is the cube of the period.*

In next section, we will introduce the functional setting for the solution spaces, with the periodic boundary condition. Finally, we will study the stability of the solutions of the stationary Stokes equations on R^n.

2. Functional spaces for the solution space

For the functional spaces of the solutions of the Stokes equations we will use the Lebesque space $L^2(R^n)$ with the periodic boundary condition. For the Sobolev space of functions which are in $L^2(\Omega)$, with all their derivatives of order $\leq m$, we set by $H^m(\Omega)$. And with the inner product and the norm,

$$(\mathbf{u}, \mathbf{v})_m = \sum_{[\alpha] \leq m} (D^\alpha \mathbf{u}, D^\alpha \mathbf{u}) \quad \text{and} \quad |\mathbf{u}|_m = [(\mathbf{u}, \mathbf{u})_m]^{1/2},$$

$H^m(\Omega)$ is a Hilbert space. We also consider the subset $H_p^m(\Omega)$ of $H^m(\Omega)$, the space of periodic functions with the cube Ω of period :

$$\mathbf{u}(\mathbf{x} + L_i \mathbf{e}_i) = \mathbf{u}(\mathbf{x}) \quad \text{for all} \quad i = 1, \dots, n. \tag{2.1}$$

For $m = 0$, $H_p^0(\Omega)$ means $L^2(\Omega)$.

Then, $H_p^m(\Omega)$ is also a Hilbert space and the functions in $H_p^m(\Omega)$ are characterized by their Fourier series expansion

(2.2)

$$H_p^m(\Omega) = \left\{ \mathbf{u} \ : \ \mathbf{u} = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}} e^{2i\pi \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \ \bar{c}_{\mathbf{k}} = c_{-\mathbf{k}}, \ \sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^{2m} |c_{\mathbf{k}}|^2 < \infty \right\},$$

where $\frac{\mathbf{x}}{L} = (\frac{\mathbf{x}_1}{L_1}, \cdots, \frac{\mathbf{x}_n}{L_n})$ and $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \cdots, \frac{\mathbf{k}_n}{L_n})$. We also denote

$$\dot{H}_p^m(\Omega) = \left\{ \mathbf{u} \in H_p^m(\Omega) \text{ of type (2.2)} \ : \ c_0 = 0 \right\}. \tag{2.3}$$

Now, one note that $\dot{H}_p^m(\Omega)$ is a Hilbert space with the norm $\left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^{2m} |c_{\mathbf{k}}|^2\right]^{1/2}$, and $\dot{H}_p^{-m}(\Omega)$ is the dual space of $\dot{H}_p^m(\Omega)$ for all $m \in \mathbb{N}$. Now, we introduce an new Hilbert space, $\mathbf{H}_p^m(\Omega) = \{H_p^m(\Omega)\}^n$ with the inner product and the norm,

$$((\mathbf{u}, \mathbf{v})) = \sum_{i=1}^{n} \left(\frac{\partial \mathbf{u}}{\partial x_i}, \frac{\partial \mathbf{v}}{\partial x_i} \right), \qquad \|\mathbf{u}\| = \{((\mathbf{u}, \mathbf{u}))\}^{1/2}.$$

And for the solution spaces, we will use the following functional spaces,

$$\mathbf{V} = \{ \mathbf{u} \in \mathbf{H}_p^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \ \text{ in } \ R^n \}, \tag{2.4}$$

$$\mathbf{H} = \{ \mathbf{u} \in \mathbf{H}_p^0(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \ \text{ in } \ R^n \}, \tag{2.5}$$

where **V** is also a Hilbert space with this norm. More about the functional spaces for the solutions with the periodic boundary condition can be found in [16].

The stationary Stokes problem (1.4) with the periodic boundary condition (1.5) have the following; *Given* $\mathbf{f} \in \dot{\mathbf{H}}_p^0(\Omega)$ *or* $\dot{\mathbf{H}}_p^{-1}(\Omega)$*, find* $\mathbf{u} \in \dot{\mathbf{H}}_p^1(\Omega)$ *and* $p \in L^2(\Omega)$ *such that*

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \ \ \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \ \ \Omega. \tag{2.6}$$

For the Fourier expansions of $\mathbf{u}$, p and $\mathbf{f}$, we set as the following :

$$\mathbf{u} = \sum_{\mathbf{k} \in \mathbb{Z}^n} u_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \qquad p = \sum_{\mathbf{k} \in \mathbb{Z}^n} p_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \qquad \mathbf{f} = \sum_{\mathbf{k} \in \mathbb{Z}^n} f_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}},$$

where $\frac{\mathbf{x}}{L} = (\frac{\mathbf{x}_1}{L_1}, \cdots, \frac{\mathbf{x}_n}{L_n})$.

Now, by (2.6), for every $\mathbf{k} \neq 0$ one obtain

$$4\pi^2 \left| \frac{\mathbf{k}}{L} \right|^2 u_{\mathbf{k}} + 2\pi i \frac{\mathbf{k}}{L} p_{\mathbf{k}} = f_{\mathbf{k}} \tag{2.7}$$

and

$$\frac{\mathbf{k}}{L} \cdot u_{\mathbf{k}} = 0, \tag{2.8}$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \cdots, \frac{\mathbf{k}_n}{L_n})$.

Taking the scalar product of (2.7) with $\frac{\mathbf{k}}{L}$ and using (2.8) we find

$$p_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}}{2\pi i |\frac{\mathbf{k}}{L}|^2} \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \mathbf{k} \neq 0 \tag{2.9}$$

and by (2.7) we have

$$u_{\mathbf{k}} = \frac{1}{4\pi^2 \left|\frac{\mathbf{k}}{L}\right|^2} \left(f_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right) \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \mathbf{k} \neq 0. \tag{2.10}$$

3. The stability of the solutions

Now, we will study the stability of the solutions for the n-dimensional stationary Stokes equations with the periodic boundary condition. The existence of the solutions for the n-dimensional stationary Stokes equations with the periodic boundary condition is very well known. With assumption of the existence of the solution, we will study the various stability of the solutions.

THEOREM 3.1. *Let the function* $\mathbf{v} \in \dot{\mathbf{H}}_p^2(\Omega)$ *and* $q \in \dot{\mathbf{H}}_p^1(\Omega)$ *satisfy the equations*

$$-\Delta \mathbf{v} + \nabla q - \mathbf{f} = \mathbf{g} \ \ in \ \ \Omega, \qquad \nabla \cdot \mathbf{v} = 0 \ \ in \ \ \Omega, \tag{3.1}$$

where $\|\mathbf{g}\|_{L^2} \leq \varepsilon$ *and* $\mathbf{f}, \mathbf{g} \in \dot{\mathbf{H}}_p^0(\Omega)$. *Then there exist* $\mathbf{u} \in \dot{\mathbf{H}}_p^2(\Omega)$ *and* $p \in \dot{\mathbf{H}}_p^1(\Omega)$ *satisfying*

$$-\Delta \mathbf{u} + \nabla p - \mathbf{f} = 0 \ \ in \ \ \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \ \ in \ \ \Omega \tag{3.2}$$

such that

$$\|\mathbf{u} - \mathbf{v}\|_{H^i} \leq K_i \|\mathbf{g}\|_{L^2} \leq K_i \varepsilon \quad for \quad i = 0, 1, 2, \tag{3.3}$$

$$\|p - q\|_{H^i} \leq M_i \|\mathbf{g}\|_{L^2} \leq M_i \varepsilon \quad for \quad i = 0, 1 \tag{3.4}$$

for some constants K_i *and* M_i *depending on* L_{max} *where* $L_{max} = \max\{L_1, \cdots, L_n\}$.

Proof. For existence of the solution $\mathbf{u} \in \dot{\mathbf{H}}_p^2(Q)$ and $p \in \dot{\mathbf{H}}_p^1(Q)$, one can have from (2.9) and (2.10). Next, to obtain (3.3) and (3.4) we denote the Fourier expansions of $\mathbf{v}$, q and $\mathbf{g}$ as the followings;

$$\mathbf{v} = \sum_{\mathbf{k} \in \mathbb{Z}^2} v_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \qquad q = \sum_{\mathbf{k} \in \mathbb{Z}^2} q_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}}, \qquad \mathbf{g} = \sum_{\mathbf{k} \in \mathbb{Z}^2} g_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \frac{\mathbf{x}}{L}},$$

where $\frac{\mathbf{x}}{L} = (\frac{\mathbf{x}_1}{L_1}, \cdots, \frac{\mathbf{x}_n}{L_n})$. By (2.9), (2.10) and (3.1) we have

$$q_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot (f_{\mathbf{k}} + g_{\mathbf{k}})}{2\pi i |\frac{\mathbf{k}}{L}|^2} \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \ \mathbf{k} \neq 0 \tag{3.5}$$

and

$$v_{\mathbf{k}} = \frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left((f_{\mathbf{k}} + g_{\mathbf{k}}) - \frac{(\frac{\mathbf{k}}{L} \cdot [f_{\mathbf{k}} + g_{\mathbf{k}}]) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right) \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \ \mathbf{k} \neq 0, \tag{3.6}$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \cdots, \frac{\mathbf{k}_n}{L_n})$. And, due to (2.9), (2.10) and (3.2) we can get

$$p_{\mathbf{k}} = \frac{\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}}{2\pi i |\frac{\mathbf{k}}{L}|^2} \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \ \mathbf{k} \neq 0 \tag{3.7}$$

and

$$u_{\mathbf{k}} = \frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left(f_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot f_{\mathbf{k}}) \frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right) \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n, \ \ \mathbf{k} \neq 0, \tag{3.8}$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \cdots, \frac{\mathbf{k}_n}{L_n})$. Therefore, from (3.5) - (3.8), we obtain the Fourier expansions of $\mathbf{u} - \mathbf{v}$ and $p - q$ as the following ;

$$(3.9) \qquad u_{\mathbf{k}} - v_{\mathbf{k}} = -\frac{1}{4\pi^2 |\frac{\mathbf{k}}{L}|^2} \left(g_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}})\frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} \right), \qquad p_{\mathbf{k}} - q_{\mathbf{k}} = -\frac{\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}}}{2\pi i |\frac{\mathbf{k}}{L}|^2},$$

where $\frac{\mathbf{k}}{L} = (\frac{\mathbf{k}_1}{L_1}, \cdots, \frac{\mathbf{k}_n}{L_n})$. Then, for $|\mathbf{k}| > 1$, from (3.9) we have

$$|u_{\mathbf{k}} - v_{\mathbf{k}}| \le \frac{L_{max}^2}{4\pi^2} |g_{\mathbf{k}}|,$$

where $L_{max} = \max\{L_1, \cdots, L_n\}$. Next, to prove for $|\mathbf{k}| = 1$, we assume $\mathbf{k} = (\mathbf{k}_1, \cdots, \mathbf{k}_n)$ as $\mathbf{k}_i = 0$ for all $i \neq j$ and $\mathbf{k}_j = 1$ or -1. Then for $g_{\mathbf{k}} = (g_{\mathbf{k}}^1, \cdots, g_{\mathbf{k}}^n)$, we have

$$g_{\mathbf{k}} - \frac{(\frac{\mathbf{k}}{L} \cdot g_{\mathbf{k}})\frac{\mathbf{k}}{L}}{|\frac{\mathbf{k}}{L}|^2} = w_{\mathbf{k}},$$

where $w_{\mathbf{k}}^i = g_{\mathbf{k}}^i$ for all $i \neq j$, and $w_{\mathbf{k}}^j = 0$. Therefore, we have

$$(3.10) \qquad |u_{\mathbf{k}} - v_{\mathbf{k}}| \le \frac{L_{max}^2}{4\pi^2} |g_{\mathbf{k}}|.$$

Hence, we have

$$(3.11) \qquad \|\mathbf{u} - \mathbf{v}\|_{L^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \le \frac{L_{max}^2}{4\pi^2} \|\mathbf{g}\|_{L^2} \le K_1 \varepsilon$$

and similarly we obtain

$$(3.12) \qquad \|p - q\|_{L^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |p_{\mathbf{k}} - q_{\mathbf{k}}|^2 \right]^{1/2} \le \frac{L_{max}}{2\pi} \|\mathbf{g}\|_{L^2} \le M_1 \varepsilon.$$

Also, for H^1-norm, by (3.9) we get

$$(3.13) \qquad \|\mathbf{u} - \mathbf{v}\|_{H^1} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^2 |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \le \frac{L_{max}}{2\pi^2} \|\mathbf{g}\|_{L^2} \le K_2 \varepsilon$$

and

$$(3.14) \qquad \|p - q\|_{H^1} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^2 |p_{\mathbf{k}} - q_{\mathbf{k}}|^2 \right]^{1/2} \le \frac{1}{2\pi} \|\mathbf{g}\|_{L^2} \le M_2 \varepsilon.$$

Similarly, for H^2-norm, we have

$$(3.15) \qquad \|\mathbf{u} - \mathbf{v}\|_{H^2} = \left[\sum_{\mathbf{k} \in \mathbb{Z}^n} |\frac{\mathbf{k}}{L}|^4 |u_{\mathbf{k}} - v_{\mathbf{k}}|^2 \right]^{1/2} \le \frac{1}{4\pi^2} \|\mathbf{g}\|_{L^2} \le K_3 \varepsilon.$$

Hence, by (3.11) - (3.15), we complete the proof. □

REMARK 3.2. We will see that the constant in (3.10) is optimal. Assume that $L_{max} = \max\{L_1, \cdots, L_n\} = L_j$. Consider the function g as $g_{\mathbf{k}} = 0$ if $\mathbf{k} \neq (0, \cdots, 0, 1, 0, \cdots, 0)$, and

$$g_{\mathbf{k}} = (\frac{\epsilon}{\sqrt{n-1}}, \cdots, \frac{\epsilon}{\sqrt{n-1}}, 0, \frac{\epsilon}{\sqrt{n-1}}, \cdots, \frac{\epsilon}{\sqrt{n-1}})$$

if $\mathbf{k} = (0, \cdots, 0, 1, 0, \cdots, 0)$ which means that every component equal to zero except j-th component. Then the inequality (3.10) becomes to the equality. Therefore, our Hyers-Ulam constant is **optimal**.

COROLLARY 3.3. *Assume that the function* $\mathbf{v} \in \dot{\mathbf{H}}_p^2(\Omega)$ *and* $q \in \dot{\mathbf{H}}_p^1(\Omega)$ *satisfy the equations*

$$-\Delta \mathbf{v} + \nabla q - \mathbf{f} = \mathbf{g} \quad in \quad \Omega, \qquad \nabla \cdot \mathbf{v} = 0 \quad in \quad \Omega, \tag{3.16}$$

where $\mathbf{f} \in \dot{\mathbf{H}}_p^0(\Omega)$ *and* $\mathbf{g} \in \mathbf{H}$ *with* $\|\mathbf{g}\|_{L^2} \leq \varepsilon$. *Then there exist* $\mathbf{u} \in \dot{\mathbf{H}}_p^2(\Omega)$ *and* $p \in \dot{\mathbf{H}}_p^1(\Omega)$ *satisfying*

$$-\Delta \mathbf{u} + \nabla p - \mathbf{f} = 0 \quad in \quad \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad in \quad \Omega. \tag{3.17}$$

such that

$$\|\mathbf{u} - \mathbf{v}\|_{H^i} \leq K_i \|\mathbf{g}\|_{L^2} \leq K_i \varepsilon \quad for \quad i = 0, 1, 2, \tag{3.18}$$

$$\|p - q\|_{H^i} = 0 \quad for \quad i = 0, 1, \tag{3.19}$$

for some constants K_i *depending on* L_{max} *where* $L_{max} = \max\{L_1, \cdots, L_n\}$.

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References

[1] E. Gselmann, *Stability properties in some classes of second order partial differential equations*, Results. Math., **65** (2014), 95-103.
[2] B. Hegyi and S.-M. Jung, *On the stability of Laplace's equation*, Appl. Math. Lett., **26** (2013), 549-552.
[3] S.-M. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett., **17** (2004), 1135-1140.
[4] S.-M. Jung, *Hyers-Ulam stability of linear differential equations of first order, II*, Appl. Math. Lett., **19** (2006), 854-858.
[5] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.
[6] S.-M. Jung, *On the stability of wave equation*, Abstr. Appl. Anal., **2013** (2013), Article ID 910565, 6 pages.
[7] S.-M. Jung and K.-S. Lee, *Hyers-Ulam stability of first order linear partial differential equations with constant coefficients*, Math. Inequal. Appl., **10** (2007), 261-266.
[8] S.-M. Jung and J. Roh, *The linear differential equations with complex constant coefficients and Schrödinger equations*, Appl. Math. Lett., **66** (2017), 23-29.

[9] S.-M. Jung and J. Roh, *Hyers–Ulam stability of the time independent Schrödinger equations*, Appl. Math. Lett., **74** (2017), 147-153.

[10] S.-M. Jung and J. Roh, *Approximation Property of the Stationary Stokes Equations with the Periodic Boundary Condition*, Journal of function spaces, **2018** (2018), 1-5.

[11] Y. Li and Y. Shen, *Hyers-Ulam stability of linear differential equations of second order*, Appl. Math. Lett., **23** (2010), 306-309.

[12] M. Obłoza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat., **13** (1993), 259-270.

[13] M. Obłoza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat., **14** (1997), 141-146.

[14] D. Popa and I. Raşa, *On the Hyers-Ulam stability of the linear differential equation*, J. Math. Anal. Appl., **381** (2011), 530-537.

[15] A. Prástaro and Th. M. Rassias, *Ulam stability in geometry of PDE's*, Nonlinear Funct. Anal. Appl., **8** (2003), 259-278.

[16] Roger Temam, *Naver-Stokes Equations and Nonlinear Functional Analysis*, Society for Industrial and Applied Mathematics, 1983.

[17] G. Wang, M. Zhou and L. Sun, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett., **21** (2008), 1024-1028.

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